4 Decision Making for a Single Sample

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter, you should be able to do the following:

1. Perform hypothesis tests and construct confidence intervals on the mean of a normal distribution.
2. Perform hypothesis tests and construct confidence intervals on the variance of a normal distribution.
3. Perform hypothesis tests and construct confidence intervals on a population proportion.
4. Compute power and type II error, and make sample-size selection decisions for hypothesis tests and confidence intervals.
5. Explain and use the relationship between confidence intervals and hypothesis tests.
6. Construct a prediction interval for a future observation.
7. Construct a tolerance interval for a normal population.
8. Explain the difference between confidence intervals, prediction intervals, and tolerance intervals.
9. Use the chi-square goodness-of-fit test to check distributional assumptions.

4-1 Statistical Inference

- The field of statistical inference consists of those methods used to make decisions or draw conclusions about a **population**.

- These methods utilize the information contained in a **sample** from the population in drawing conclusions.

4-1 Statistical Inference

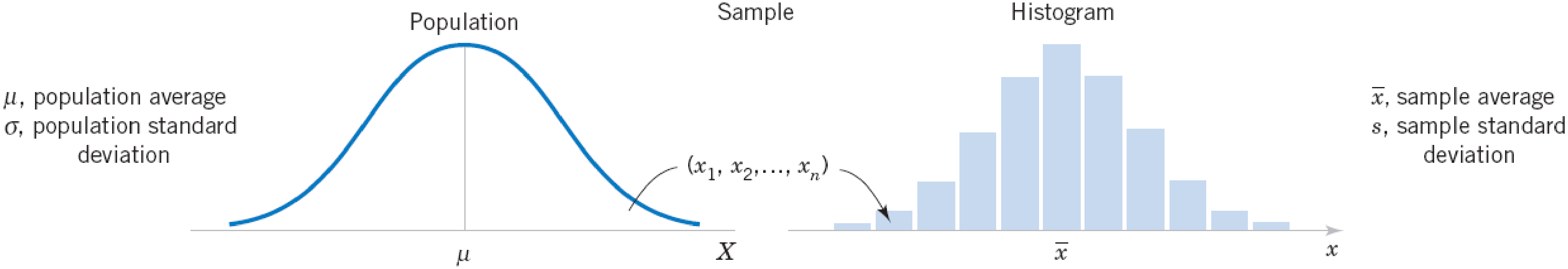


Figure 4-1 Relationship between a population and a sample.

4-2 Point Estimation

A **point estimate** of some population parameter θ is a single numerical value $\hat{\theta}$ of a statistic $\hat{\Theta}$.

Unknown Parameter θ	Statistic $\hat{\Theta}$	Point Estimate $\hat{\theta}$
μ	$\overline{X} = \frac{\Sigma X_i}{n}$	$\overline{x}$
σ^2	$S^2 = \frac{\Sigma(X_i - \overline{X})^2}{n-1}$	s^2
p	$\hat{P} = \frac{X}{n}$	$\hat{p}$
$\mu_1 - \mu_2$	$\overline{X}_1 - \overline{X}_2 = \frac{\Sigma X_{1i}}{n_1} - \frac{\Sigma X_{2i}}{n_2}$	$\overline{x}_1 - \overline{x}_2$
$p_1 - p_2$	$\hat{P}_1 - \hat{P}_2 = \frac{X_1}{n_1} - \frac{X_2}{n_2}$	$\hat{p}_1 - \hat{p}_2$

4-2 Point Estimation

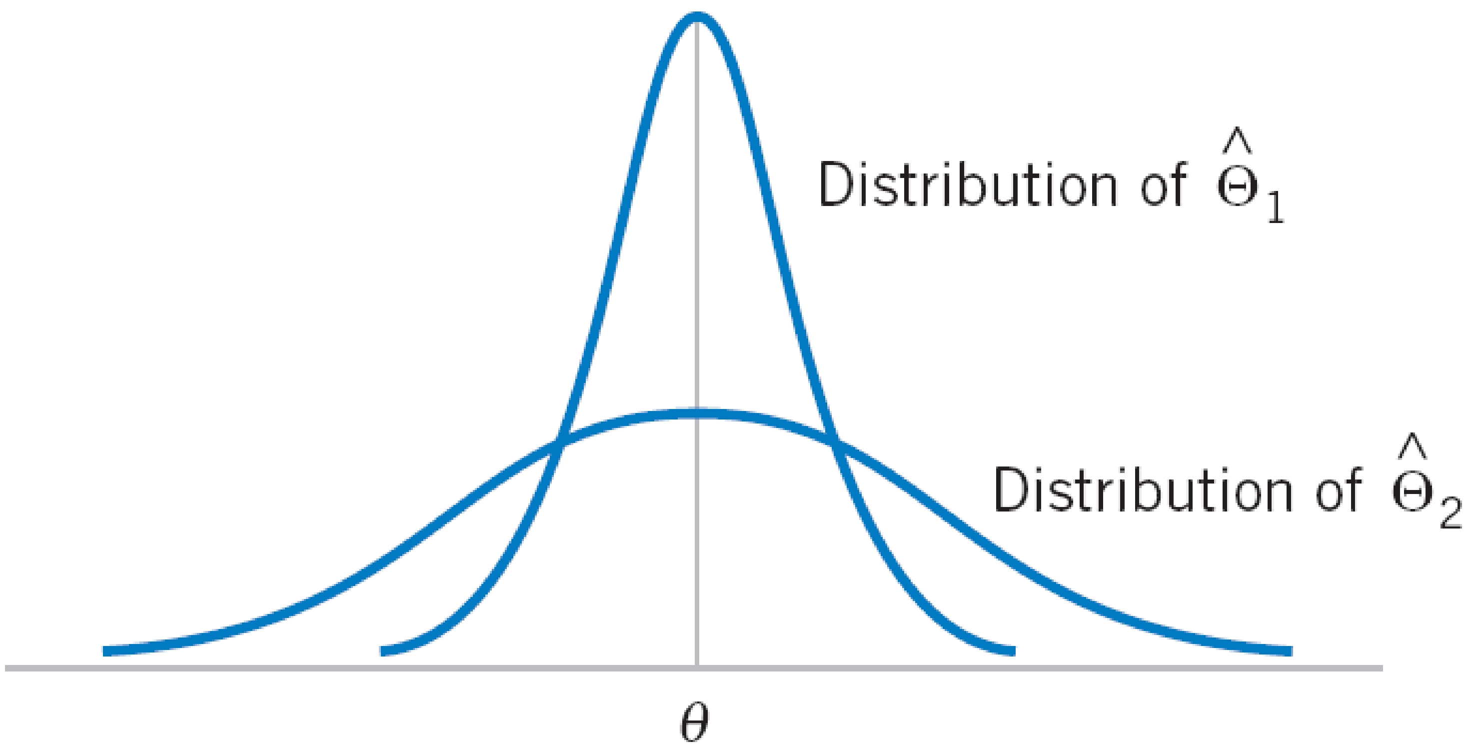


Figure 4-2 The sampling distributions of two unbiased estimators $\hat{\Theta}_1$ and $\hat{\Theta}_2$.

4-2 Point Estimation

If we consider all unbiased estimators of θ, the one with the smallest variance is called the **minimum variance unbiased estimator** (MVUE).

The **mean square error** of an estimator $\hat{\Theta}$ of the parameter θ is defined as

$$MSE(\hat{\Theta}) = E(\hat{\Theta} - \theta)^2 \qquad (4\text{-}3)$$

The **standard error** of a statistic is the standard deviation of its sampling distribution. If the standard error involves unknown parameters whose values can be estimated, substitution of these estimates into the standard error results in an **estimated standard error.**

4-3 Hypothesis Testing

4-3.1 Statistical Hypotheses

We like to think of statistical hypothesis testing as the data analysis stage of a **comparative experiment**, in which the engineer is interested, for example, in comparing the mean of a population to a specified value (e.g. mean pull strength).

A **statistical hypothesis** is a statement about the parameters of one or more populations.

4-3 Hypothesis Testing

4-3.1 Statistical Hypotheses

For example, suppose that we are interested in the burning rate of a solid propellant used to power aircrew escape systems.

- Now burning rate is a random variable that can be described by a probability distribution.
- Suppose that our interest focuses on the **mean** burning rate (a parameter of this distribution).
- Specifically, we are interested in deciding whether or not the mean burning rate is 50 centimeters per second.

4-3 Hypothesis Testing

4-3.1 Statistical Hypotheses

Two-sided Alternative Hypothesis

$$H_0\colon \mu = 50 \text{ cm/s}$$
$$H_1\colon \mu \neq 50 \text{ cm/s}$$

One-sided Alternative Hypotheses

$H_0\colon \mu = 50 \text{ cm/s} \quad H_1\colon \mu < 50 \text{ cm/s}$ or $H_0\colon \mu = 50 \text{ cm/s} \quad H_1\colon \mu > 50 \text{ cm/s}$

4-3 Hypothesis Testing

4-3.1 Statistical Hypotheses

Test of a Hypothesis

• A procedure leading to a decision about a particular hypothesis

• Hypothesis-testing procedures rely on using the information in a **random sample from the population of interest**.

• If this information is *consistent* with the hypothesis, then we will conclude that the hypothesis is **true**; if this information is *inconsistent* with the hypothesis, we will conclude that the hypothesis is **false**.

4-3 Hypothesis Testing

4-3.2 Testing Statistical Hypotheses

$$H_0: \mu = 50 \text{ cm/s}$$
$$H_1: \mu \neq 50 \text{ cm/s}$$

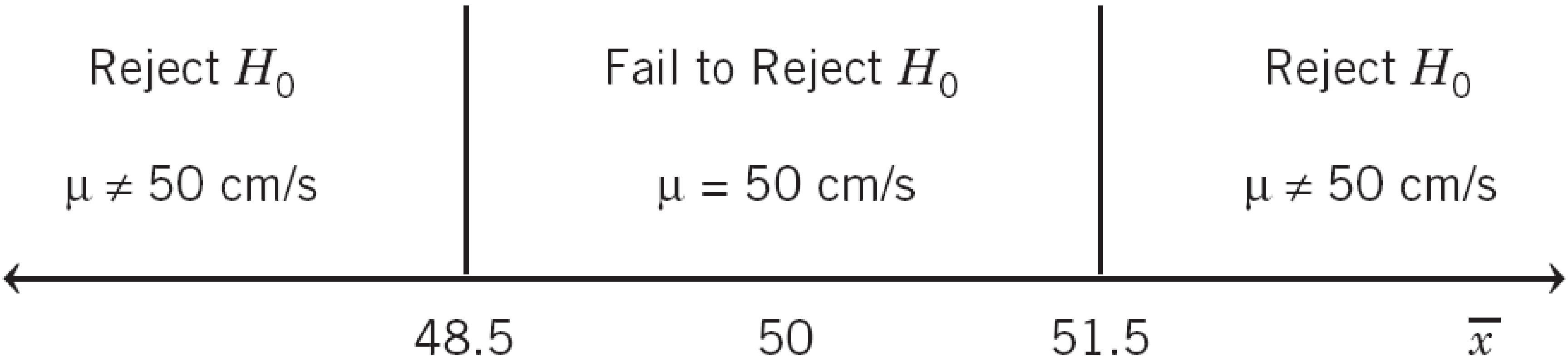


Figure 4-3 Decision criteria for testing $H_0: \mu = 50$ cm/s versus $H_1: \mu \neq 50$ cm/s.

4-3 Hypothesis Testing

4-3.2 Testing Statistical Hypotheses

Rejecting the null hypothesis H_0 when it is true is defined as a **type I error.**

Failing to reject the null hypothesis when it is false is defined as a **type II error.**

4-3 Hypothesis Testing

4-3.2 Testing Statistical Hypotheses

Table 4-1 Decisions in Hypothesis Testing

Decision	H_0 Is True	H_0 Is False
Fail to reject H_0	No error	Type II error
Reject H_0	Type I error	No error

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true})$$

Sometimes the type I error probability is called the **significance level**, or the **α-error**, or the **size** of the test.

4-3 Hypothesis Testing

4-3.2 Testing Statistical Hypotheses

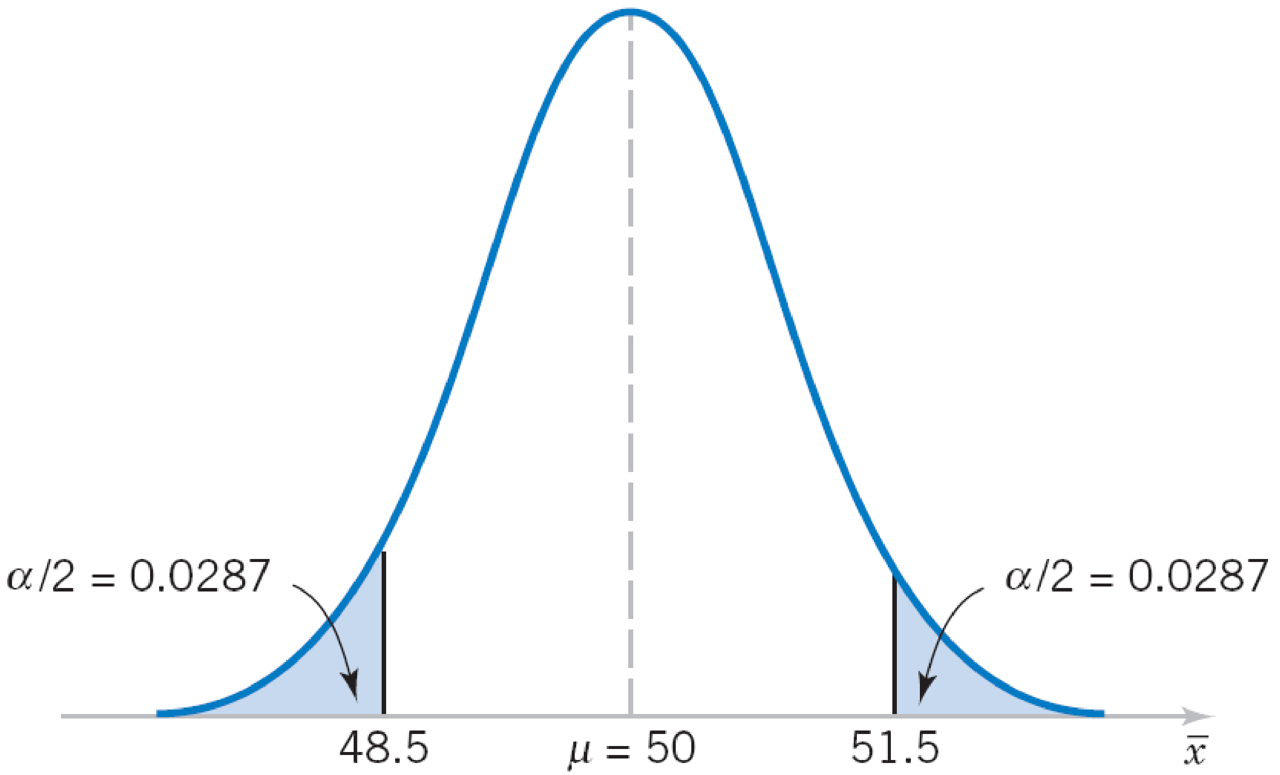


Figure 4-4 The critical region for H_0: $\mu = 50$ versus H_1: $\mu \neq 50$ and $n = 10$.

4-3 Hypothesis Testing

4-3.2 Testing Statistical Hypotheses

$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_0 \text{ is false})$$

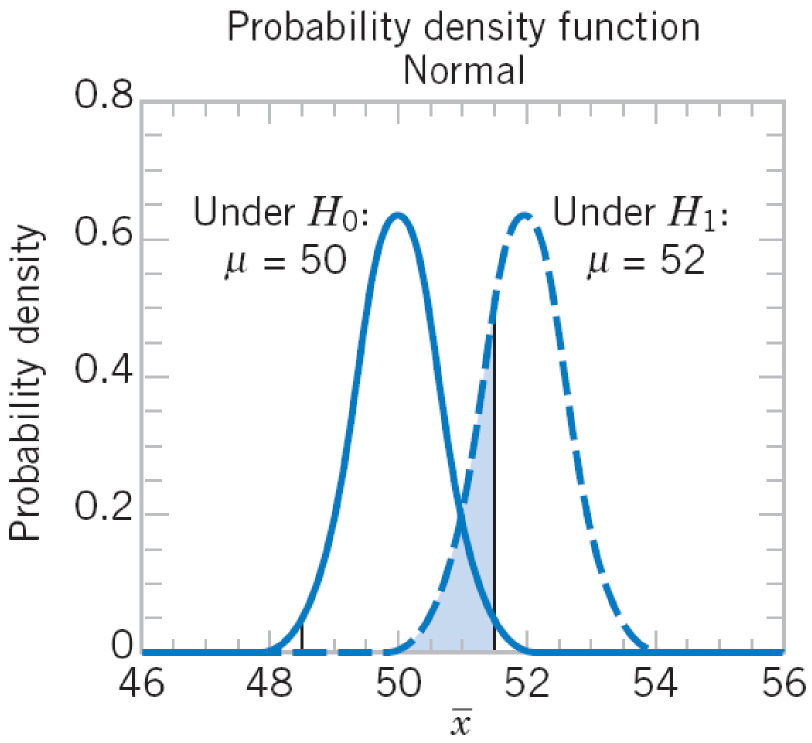


Figure 4-5 The probability of type II error when $\mu = 52$ and $n = 10$.

4-3 Hypothesis Testing

4-3.2 Testing Statistical Hypotheses

$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_0 \text{ is false})$$

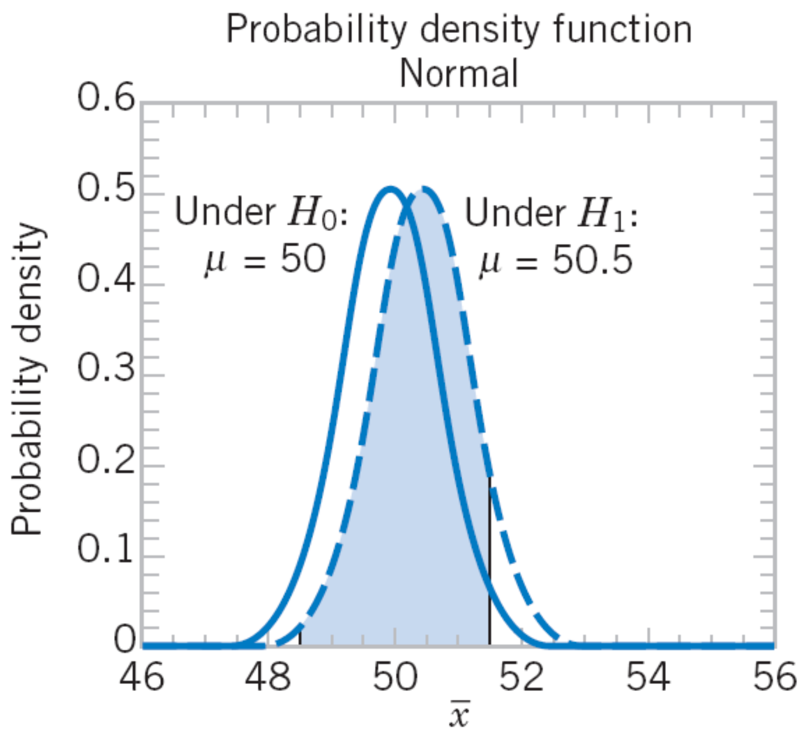


Figure 4-6 The probability of type II error when $\mu = 50.5$ and $n = 10$.

4-3 Hypothesis Testing

4-3.2 Testing Statistical Hypotheses

$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_0 \text{ is false})$$

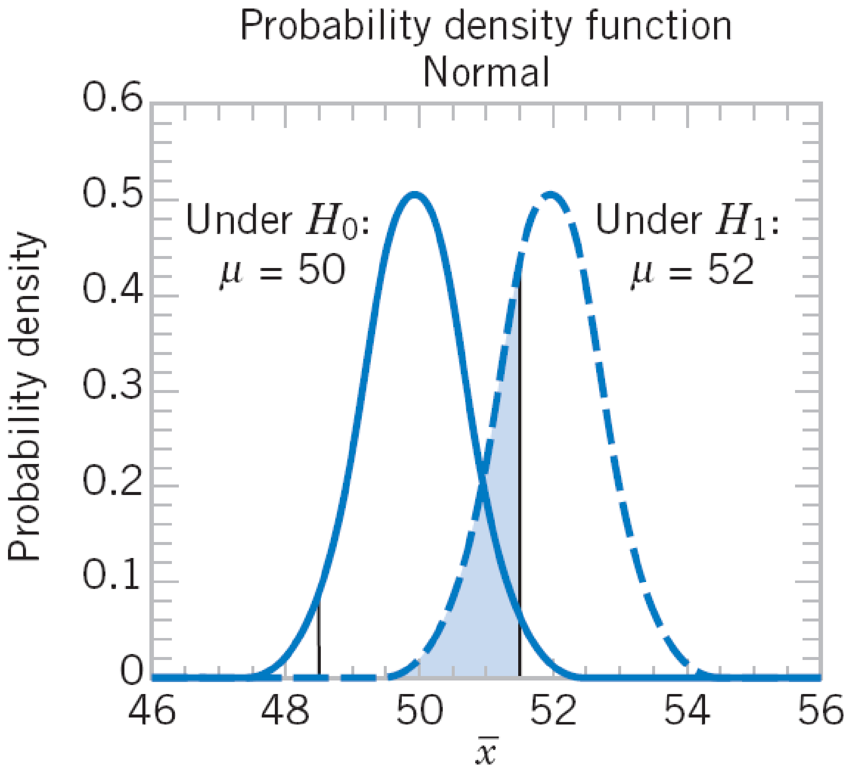


Figure 4-7 The probability of type II error when $\mu = 52$ and $n = 16$.

4-3 Hypothesis Testing

4-3.2 Testing Statistical Hypotheses

Fail to Reject H_0 When	Sample Size	α	β at $\mu = 52$	β at $\mu = 50.5$
$48.5 < \overline{x} < 51.5$	10	0.0574	0.2643	0.8923
$48 < \overline{x} < 52$	10	0.0114	0.5000	0.9705
$48.5 < \overline{x} < 51.5$	16	0.0164	0.2119	0.9445
$48 < \overline{x} < 52$	16	0.0014	0.5000	0.9918

4-3 Hypothesis Testing

4-3.2 Testing Statistical Hypotheses

The **power** of a statistical test is the probability of rejecting the null hypothesis H_0 when the alternative hypothesis is true.

- The power is computed as 1 - β, and power can be interpreted as *the probability of correctly rejecting a false null hypothesis*. We often compare statistical tests by comparing their **power** properties.

- For example, consider the propellant burning rate problem when we are testing $H_0 : \mu = 50$ centimeters per second against $H_1 : \mu$ not equal 50 centimeters per second . Suppose that the true value of the mean is $\mu = 52$. When $n = 10$, we found that $\beta = 0.2643$, so the power of this test is $1 - \beta = 1 - 0.2643 = 0.7357$ when $\mu = 52$.

4-3 Hypothesis Testing

4-3.3 One-Sided and Two-Sided Hypotheses

Two-Sided Test:

$$H_0\colon \mu = \mu_0$$
$$H_1\colon \mu \neq \mu_0$$

One-Sided Tests:

$$H_0\colon \mu = \mu_0 \qquad\qquad H_0\colon \mu = \mu_0$$
$$\text{or}$$
$$H_1\colon \mu > \mu_0 \qquad\qquad H_1\colon \mu < \mu_0$$

4-3 Hypothesis Testing

4-3.3 P-Values in Hypothesis Testing

The ***P*-value** is the smallest level of significance that would lead to rejection of the null hypothesis H_0.

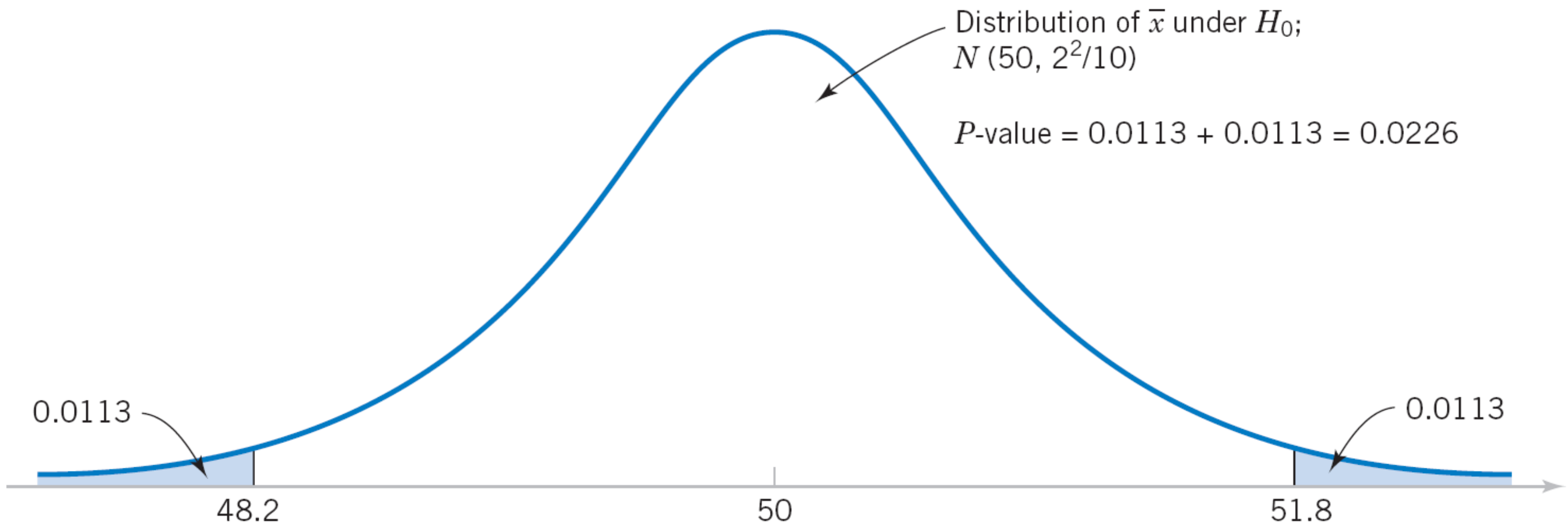


Figure 4-8 Calculating the *P*-value for the propellant burning rate problem.

4-3 Hypothesis Testing

4-3.5 General Procedure for Hypothesis Testing

1. **Parameter of interest:** From the problem context, identify the parameter of interest.
2. **Null hypothesis, H_0:** State the null hypothesis, H_0.
3. **Alternative hypothesis, H_1:** Specify an appropriate alternative hypothesis, H_1.
4. **Test statistic:** State an appropriate test statistic.
5. **Reject H_0 if:** Define the criteria that will lead to rejection of H_0.
6. **Computations:** Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute that value.
7. **Conclusions:** Decide whether or not H_0 should be rejected and report that in the problem context. This could involve computing a P-value or comparing the test statistic to a set of critical values.

Steps 1–4 should be completed prior to examination of the sample data.

4-4 Inference on the Mean of a Population, Variance Known

Assumptions

1. $X_1, X_2, \ldots, X_n$ is a random sample of size n from a population.
2. The population is normally distributed, or if it is not, the conditions of the central limit theorem apply.

Under the previous assumptions, the quantity

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \tag{4-11}$$

has a standard normal distribution, $N(0, 1)$.

4-4 Inference on the Mean of a Population, Variance Known

4-4.1 Hypothesis Testing on the Mean

We wish to test:

$$H_0: \mu = \mu_0$$
$$H_1: \mu \neq \mu_0$$

The **test statistic** is:

$$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \qquad (4\text{-}13)$$

4-4 Inference on the Mean of a Population, Variance Known

4-4.1 Hypothesis Testing on the Mean

Reject H_0 if the observed value of the test statistic z_0 is either:

$$z_0 > z_{\alpha/2} \quad \text{or} \quad z_0 < -z_{\alpha/2}$$

Fail to reject H_0 if

$$-z_{\alpha/2} \leq z_0 \leq z_{\alpha/2}$$

4-4 Inference on the Mean of a Population, Variance Known

4-4.1 Hypothesis Testing on the Mean

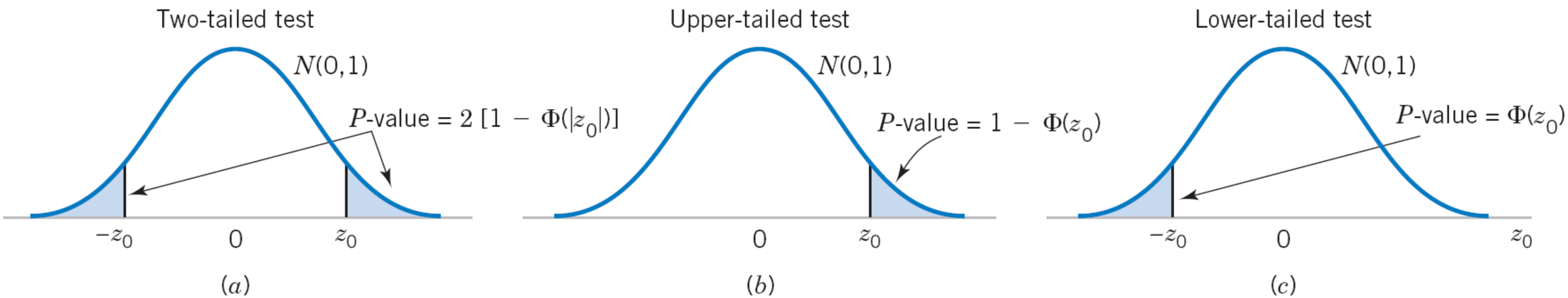


Figure 4-9 The *P*-value for a *z*-test. (*a*) The two-sided alternative H_1: $\mu \neq \mu_0$. (*b*) The one-sided alternative H_1: $\mu > \mu_0$. (*c*) The one-sided alternative H_1: $\mu < \mu_0$.

4-4 Inference on the Mean of a Population, Variance Known

4-4.1 Hypothesis Testing on the Mean

Testing Hypotheses on the Mean, Variance Known

Null hypothesis: $H_0\colon \mu = \mu_0$

Test statistic: $Z_0 = \dfrac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$

Alternative Hypotheses	***P*-Value**	**Rejection Criterion for Fixed-Level Tests**
$H_1\colon \mu \neq \mu_0$	Probability above z_0 and probability below $-z_0$, $P = 2\left[1-\Phi(\lvert z_0\rvert)\right]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1\colon \mu > \mu_0$	Probability above z_0, $P = 1-\Phi(z_0)$	$z_0 > z_\alpha$
$H_1\colon \mu < \mu_0$	Probability below z_0, $P = \Phi(z_0)$	$z_0 < -z_\alpha$

The *P*-values and critical regions for these situations are shown in Figs. 4-9 and 4-10.

4-4 Inference on the Mean of a Population, Variance Known

4-4.2 Type II Error and Choice of Sample Size

Finding The Probability of Type II Error β

Probability of a Type II Error for the Two-Sided Alternative Hypothesis on the Mean, Variance Known

$$\beta = \Phi\left(z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) - \Phi\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) \qquad (4\text{-}24)$$

4-4 Inference on the Mean of a Population, Variance Known

4-4.2 Type II Error and Choice of Sample Size

Sample Size Formulas

Sample Size for Two-Sided Alternative Hypothesis on the Mean, Variance Known

For the two-sided alternative hypothesis on the mean with variance known and significance level α, the sample size required to detect a difference between the true and hypothesized mean of δ with power at least $1 - \beta$ is

$$n \simeq \frac{(z_{\alpha/2} + z_{\beta})^2 \sigma^2}{\delta^2} \tag{4-26}$$

where

$$\delta = \mu - \mu_0$$

If n is not an integer, the convention is to always round the sample size up to the next integer.

4-4 Inference on the Mean of a Population, Variance Known

4-4.2 Type II Error and Choice of Sample Size

Sample Size Formulas

Sample Size for One-Sided Alternative Hypothesis on the Mean, Variance Known

For the one-sided alternative hypothesis on the mean with variance known and significance level α, the sample size required to detect a difference between the true and hypothesized mean of δ with power at least $1 - \beta$ is

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{\delta^2} \tag{4-27}$$

where

$$\delta = \mu - \mu_0$$

If n is not an integer, the convention is to round the sample size up to the next integer.

4-4 Inference on the Mean of a Population, Variance Known

4-4.2 Type II Error and Choice of Sample Size

EXAMPLE 4-4

Sample Size for the Propellant Burning Rate Problem

Consider the propellant burning rate problem of Example 4-3. Suppose that the analyst wishes to design the test so that if the true mean burning rate differs from 50 cm/s by as much as 1 cm/s, the test will detect this (i.e., reject H_0: $\mu = 50$) with a high probability—say, 0.90.

Solution. Note that $\sigma = 2$, $\delta = 51 - 50 = 1$, $\alpha = 0.05$, and $\beta = 0.10$. Because $z_{\alpha/2} = z_{0.025} = 1.96$ and $z_\beta = z_{0.10} = 1.28$, the sample size required to detect this departure from H_0: $\mu = 50$ is found by equation 4-26 as

$$n \simeq \frac{(z_{\alpha/2} + z_\beta)^2\sigma^2}{\delta^2} = \frac{(1.96 + 1.28)^2 2^2}{(1)^2} \simeq 42$$

The approximation is good here because $\Phi(-z_{\alpha/2} - \delta\sqrt{n}/\sigma) = \Phi(-1.96 - (1)\sqrt{42}/2) = \Phi(-5.20) \simeq 0$, which is small relative to β. ■

4-4 Inference on the Mean of a Population, Variance Known

4-4.2 Type II Error and Choice of Sample Size

Table 4-2 Minitab Computations

1-Sample *Z* Test

Testing mean = null (versus not = null)

Calculating power for mean = null + difference

Alpha = 0.05 Sigma = 2

Difference	Sample Size	Target Power	Actual Power
1	43	0.9000	0.9064

1-Sample *Z* Test

Testing mean = null (versus not = null)

Calculating power for mean = null + difference

Alpha = 0.05 Sigma = 2

Difference	Sample Size	Target Power	Actual Power
1	28	0.7500	0.7536

1-Sample *Z* Test

Testing mean = null (versus not = null)

Calculating power for mean = null + difference

Alpha = 0.05 Sigma = 2

Difference	Sample Size	Power
1	25	0.7054

4-4 Inference on the Mean of a Population, Variance Known

4-4.3 Large Sample Test

In general, if $n \geq 30$, the sample variance s^2 will be close to σ^2 for most samples, and so s can be substituted for σ in the test procedures with little harmful effect.

4-4 Inference on the Mean of a Population, Variance Known

4-4.4 Some Practical Comments on Hypothesis Testing

The Seven-Step Procedure

Only three steps are really required:

1. Specify the hypothesis (two-, upper-, or lower-tailed).
2. Specify the test statistic to be used (such as z_0).
3. Specify the criteria for rejection (typically, the value of α, or the P-value at which rejection should occur).

4-4 Inference on the Mean of a Population, Variance Known

4-4.4 Some Practical Comments on Hypothesis Testing

Statistical versus Practical Significance

Sample Size n	P-Value When $\overline{x} = 50.5$	Power (at $\alpha = 0.05$) When $\mu = 50.5$
10	0.4295	0.1241
25	0.2113	0.2396
50	0.0767	0.4239
100	0.0124	0.7054
400	5.73×10^{-7}	0.9988
1000	2.57×10^{-15}	1.0000

4-4 Inference on the Mean of a Population, Variance Known

4-4.4 Some Practical Comments on Hypothesis Testing

Statistical versus Practical Significance

Be careful when interpreting the results from hypothesis testing when the sample size is large because any small departure from the hypothesized value μ_0 will probably be detected, even when the difference is of little or no practical significance.

4-4 Inference on the Mean of a Population, Variance Known

4-4.5 Confidence Interval on the Mean

Two-sided confidence interval:

$$P(L \leq \mu \leq U) = 1 - \alpha$$

One-sided confidence intervals:

$$P(L \leq \mu) = 1 - \alpha \qquad\qquad P(\mu \leq U) = 1 - \alpha$$

Confidence coefficient: $1 - \alpha$

4-4 Inference on the Mean of a Population, Variance Known

4-4.5 Confidence Interval on the Mean

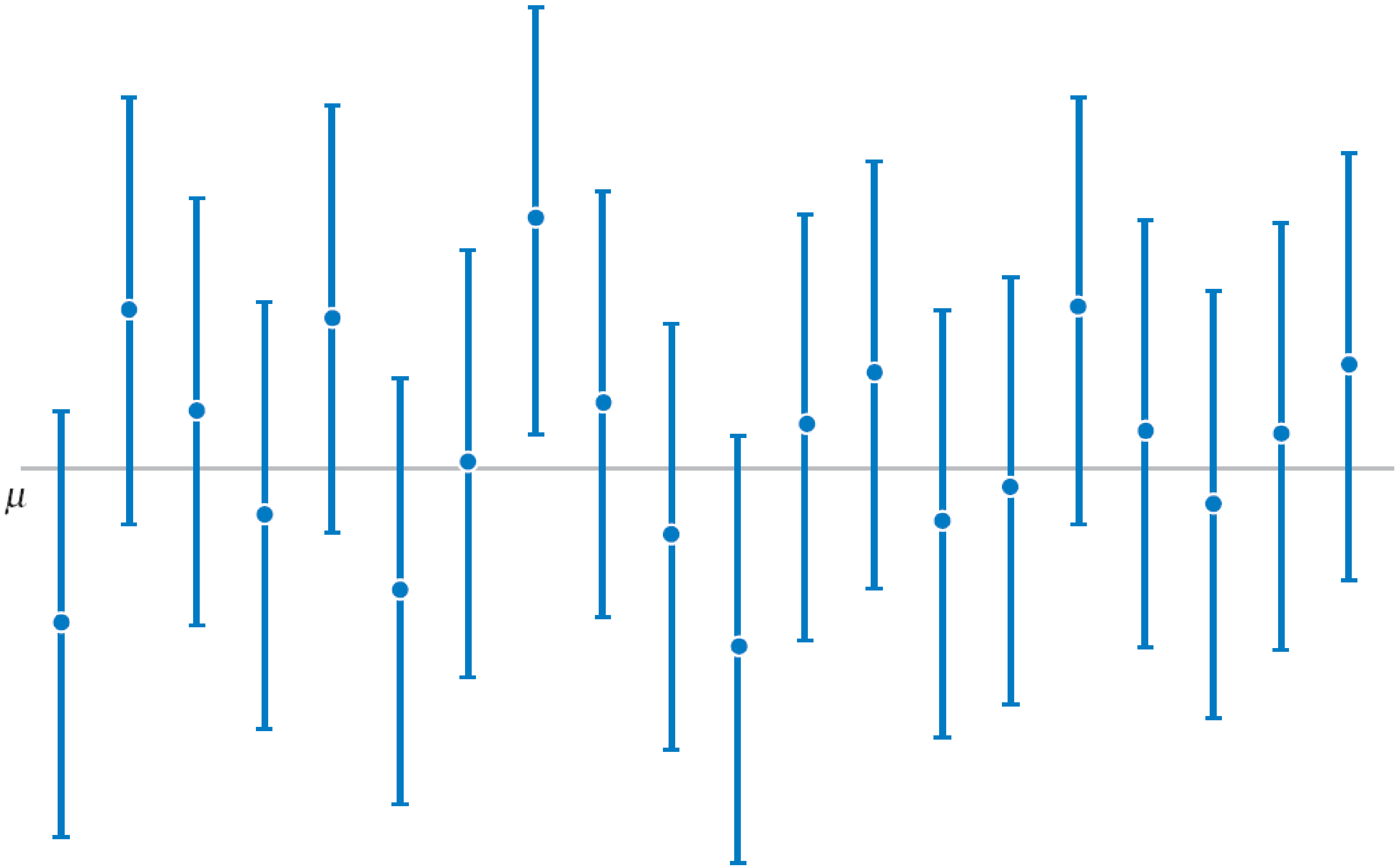


Figure 4-12 Repeated construction of a confidence interval for μ.

4-4 Inference on the Mean of a Population, Variance Known

4-4.6 Confidence Interval on the Mean

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \qquad P\{-z_{\alpha/2} \le Z \le z_{\alpha/2}\} = 1 - \alpha$$

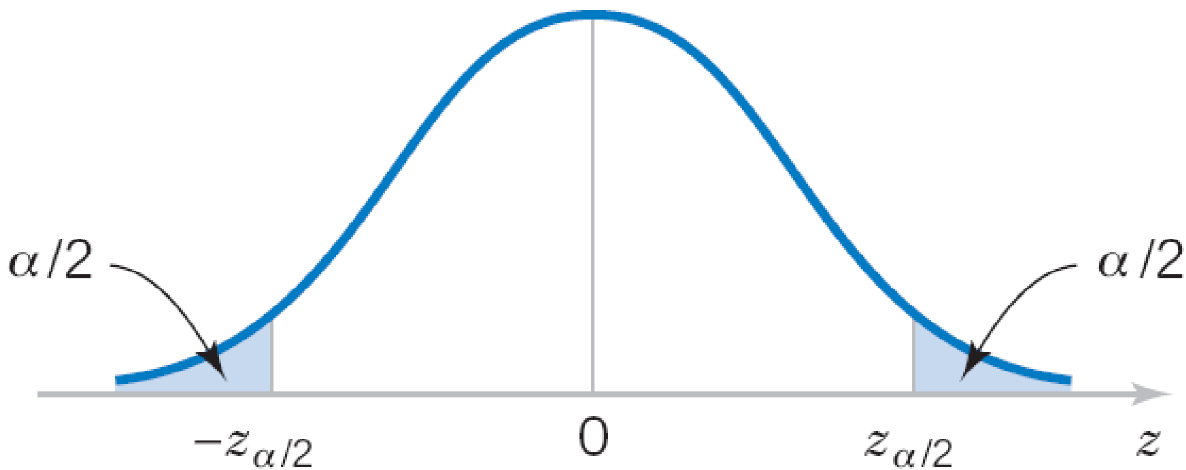


Figure 4-13 The distribution of Z.

4-4 Inference on the Mean of a Population, Variance Known

4-4.5 Confidence Interval on the Mean

Confidence Interval on the Mean, Variance Known

If $\overline{x}$ is the sample mean of a random sample of size n from a population with known variance σ^2, a $100(1 - \alpha)\%$ **confidence interval on μ** is given by

$$\overline{x} - \frac{z_{\alpha/2}\sigma}{\sqrt{n}} \leq \mu \leq \overline{x} + \frac{z_{\alpha/2}\sigma}{\sqrt{n}} \qquad (4\text{-}35)$$

where $z_{\alpha/2}$ is the upper $100\alpha/2$ percentage point and $-z_{\alpha/2}$ is the lower $100\alpha/2$ percentage point of the standard normal distribution in Appendix A Table I.

4-4 Inference on the Mean of a Population, Variance Known

4-4.5 Confidence Interval on the Mean

Relationship between Tests of Hypotheses and Confidence Intervals

If $[l,u]$ is a $100(1 - \alpha)$ percent confidence interval for the parameter, then the test of significance level α of the hypothesis

$$H_0\colon \theta = \theta_0$$

$$H_1\colon \theta \neq \theta_0$$

will lead to rejection of H_0 if and only if the hypothesized value is not in the $100(1 - \alpha)$ percent confidence interval $[l, u]$.

4-4 Inference on the Mean of a Population, Variance Known

4-4.5 Confidence Interval on the Mean

Confidence Level and Precision of Estimation

The length of the two-sided 95% confidence interval is

$$2(1.96\,\sigma/\sqrt{n}) = 3.92\,\sigma/\sqrt{n}$$

whereas the length of the two-sided 99% confidence interval is

$$2(2.58\,\sigma/\sqrt{n}) = 5.16\,\sigma/\sqrt{n}$$

4-4 Inference on the Mean of a Population, Variance Known

4-4.5 Confidence Interval on the Mean

Choice of Sample Size

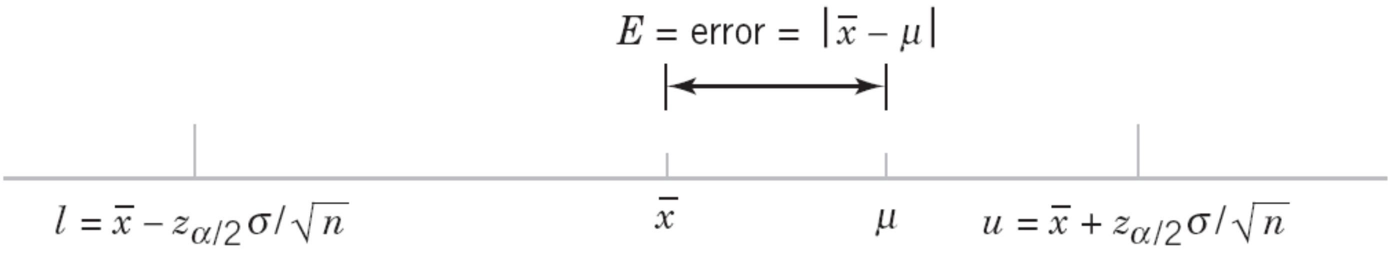


Figure 4-14 Error in estimating μ with $\bar{x}$.

Sample Size for a Specified E on the Mean, Variance Known

If $\bar{x}$ is used as an estimate of μ, we can be $100(1 - \alpha)\%$ confident that the error $|\bar{x} - \mu|$ will not exceed a specified amount E when the sample size is

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 \qquad (4\text{-}36)$$

4-4 Inference on the Mean of a Population, Variance Known

4-4.5 Confidence Interval on the Mean

Choice of Sample Size

EXAMPLE 4-6

Propellant Burning Rate

To illustrate the use of this procedure, suppose that we wanted the error in estimating the mean burning rate of the rocket propellant to be less than 1.5 cm/s, with 95% confidence. Find the required sample size.

Solution. Because $\sigma = 2$ and $z_{0.025} = 1.96$, we may find the required sample size from equation 4-36 as

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left[\frac{(1.96)2}{1.5}\right]^2 = 6.83 \cong 7$$

4-4 Inference on the Mean of a Population, Variance Known

4-4.5 Confidence Interval on the Mean

Choice of Sample Size

EXAMPLE 4-6

Note the general relationship between sample size, desired length of the confidence interval $2E$, confidence level $100(1 - \alpha)\%$, and standard deviation σ:

- As the desired length of the interval $2E$ decreases, the required sample size n increases for a fixed value of σ and specified confidence.
- As σ increases, the required sample size n increases for a fixed desired length $2E$ and specified confidence.
- As the level of confidence increases, the required sample size n increases for fixed desired length $2E$ and standard deviation σ. ■

4-4 Inference on the Mean of a Population, Variance Known

4-4.5 Confidence Interval on the Mean

One-Sided Confidence Bounds

One-Sided Confidence Bounds on the Mean, Variance Known

The $100(1-\alpha)\%$ **upper-confidence bound** for μ is

$$\mu \le u = \bar{x} + z_{\alpha}\sigma/\sqrt{n} \tag{4-37}$$

and the $100(1-\alpha)\%$ **lower-confidence bound** for μ is

$$\bar{x} - z_{\alpha}\sigma/\sqrt{n} = l \le \mu \tag{4-38}$$

4-4 Inference on the Mean of a Population, Variance Known

4-4.6 General Method for Deriving a Confidence Interval

It is easy to give a general method for finding a CI for an unknown parameter θ. Let $X_1, X_2, \ldots, X_n$ be a random sample of n observations. Suppose we can find a statistic $g(X_1, X_2, \ldots, X_n; \theta)$ with the following properties:

1. $g(X_1, X_2, \ldots, X_n; \theta)$ depends on both the sample and θ, and
2. the probability distribution of $g(X_1, X_2, \ldots, X_n; \theta)$ does not depend on θ or any other unknown parameter.

$$P[C_L \le g(X_1, X_2, \ldots, X_n; \theta) \le C_U] = 1 - \alpha$$

$$P[L(X_1, X_2, \ldots, X_n) \le \theta \le U(X_1, X_2, \ldots, X_n)] = 1 - \alpha$$

4-5 Inference on the Mean of a Population, Variance Unknown

4-5.1 Hypothesis Testing on the Mean

Let $X_1, X_2, \ldots, X_n$ be a random sample for a normal distribution with unknown mean μ and unknown variance σ^2. The quantity

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has a t distribution with $n - 1$ degrees of freedom.

4-5 Inference on the Mean of a Population, Variance Unknown

4-5.1 Hypothesis Testing on the Mean

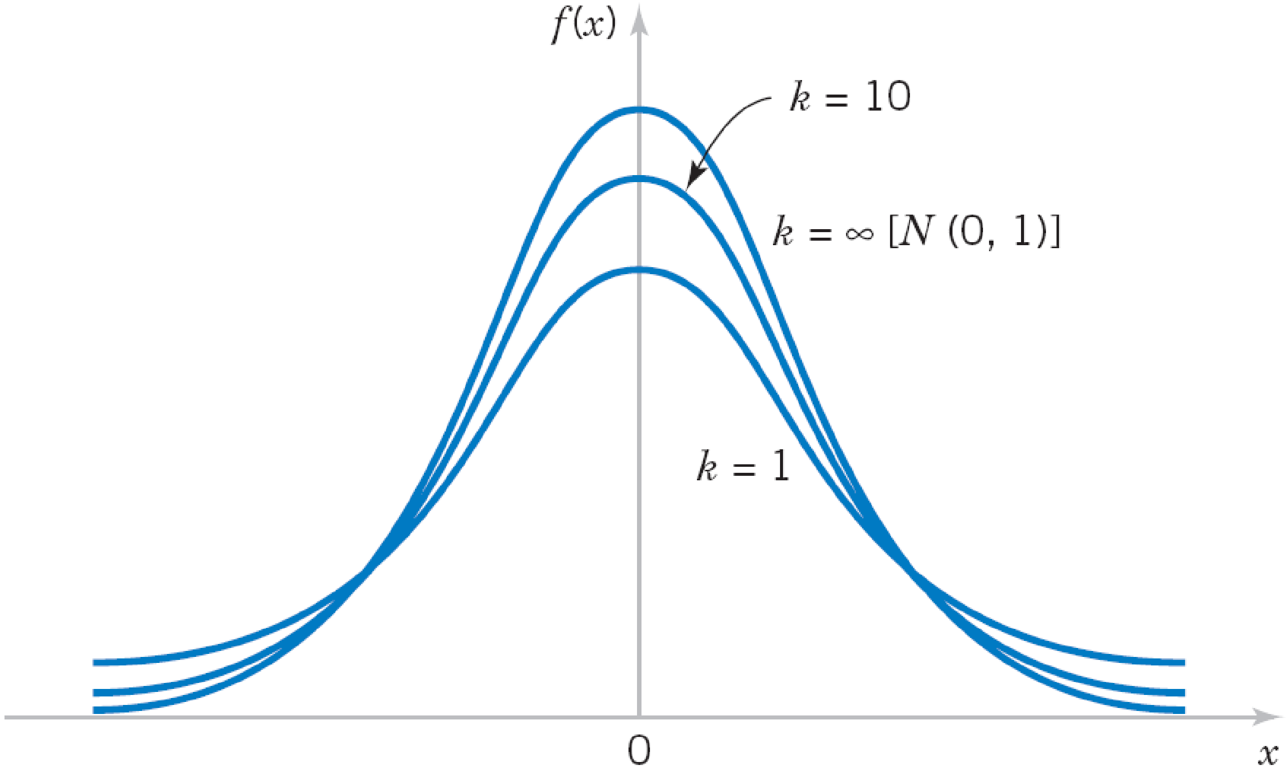


Figure 4-15 Probability density functions of several t distributions.

4-5 Inference on the Mean of a Population, Variance Unknown

4-5.1 Hypothesis Testing on the Mean

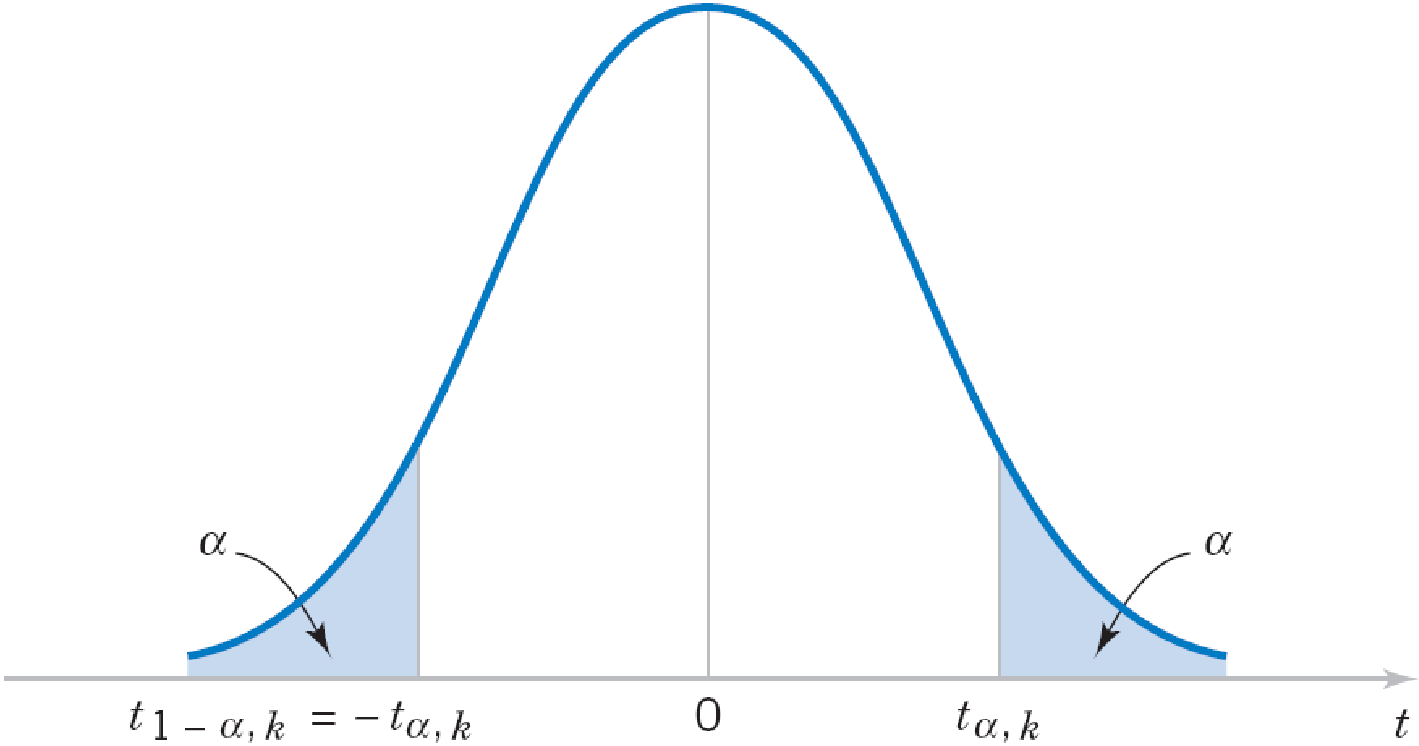


Figure 4-16 Percentage points of the t distribution.

4-5 Inference on the Mean of a Population, Variance Unknown

4-5.1 Hypothesis Testing on the Mean

Calculating the P-value

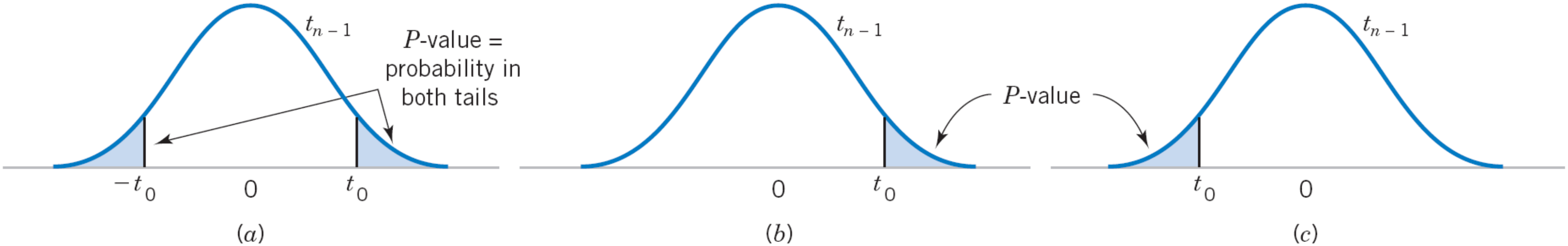


Figure 4-17 Calculating the *P*-value for a *t*-test: (*a*) H_1: $\mu \neq \mu_0$; (*b*) H_1: $\mu > \mu_0$; (*c*) H_1: $\mu < \mu_0$.

4-5 Inference on the Mean of a Population, Variance Unknown

4-5.1 Hypothesis Testing on the Mean

Testing Hypotheses on the Mean of a Normal Distribution, Variance Unknown

Null hypothesis: $H_0: \mu = \mu_0$

Test statistic: $T_0 = \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}}$

Alternative Hypotheses	***P*-Value**	**Rejection Criterion for Fixed-Level Tests**
$H_1: \mu \neq \mu_0$	Sum of the probability above t_0 and the probability below $-t_0$	$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$
$H_1: \mu > \mu_0$	Probability above t_0	$t_0 > t_{\alpha,n-1}$
$H_1: \mu < \mu_0$	Probability below t_0	$t_0 < -t_{\alpha,n-1}$

The locations of the critical regions for these situations are shown in Fig. 4-19*a*, *b*, and *c*, respectively.

4-5 Inference on the Mean of a Population, Variance Unknown

4-5.1 Hypothesis Testing on the Mean

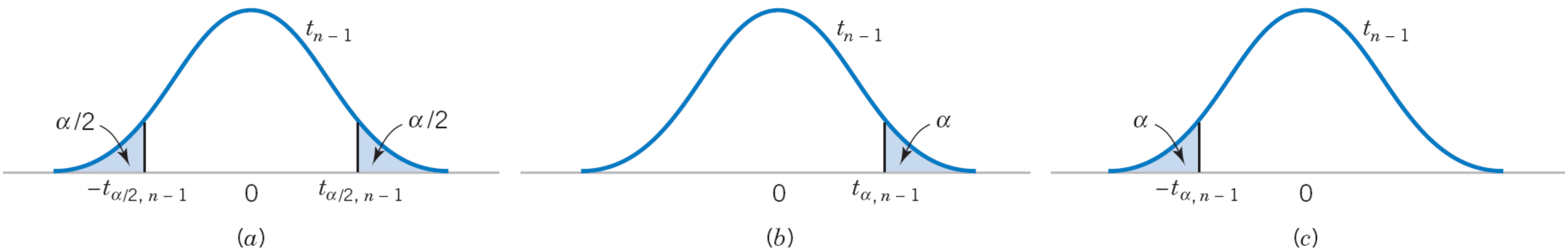
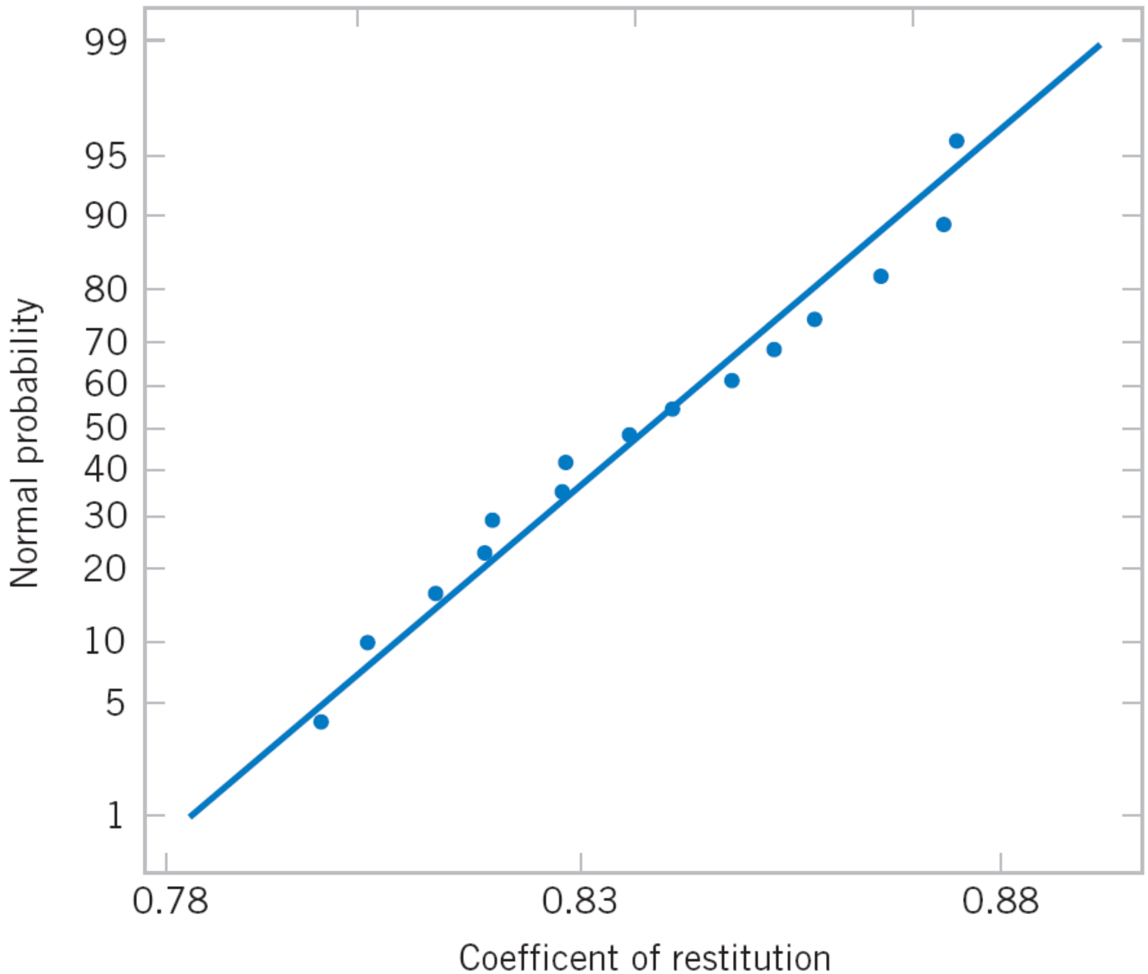


Figure 4-19 The distribution of T_0 when H_0: $\mu = \mu_0$ is true, with critical region for (a) H_1: $\mu \neq \mu_0$, (b) H_1: $\mu > \mu_0$, and (c) H_1: $\mu < \mu_0$.

4-5 Inference on the Mean of a Population, Variance Unknown

4-5.1 Hypothesis Testing on the Mean



4-5 Inference on the Mean of a Population, Variance Unknown

4-5.1 Hypothesis Testing on the Mean

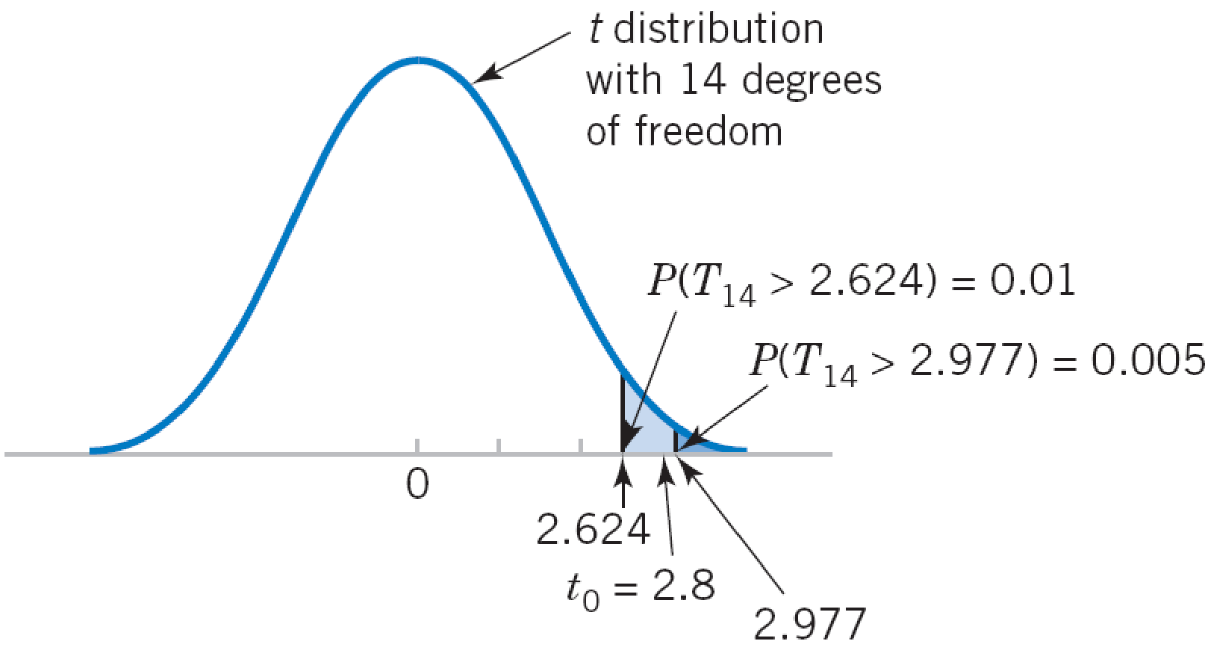


Figure 4-18 P-value for $t_0 = 2.8$ and an upper-tailed test is shown to be between 0.005 and 0.01.

4-5 Inference on the Mean of a Population, Variance Unknown

4-5.2 Type II Error and Choice of Sample Size

$$\beta = P\{-t_{\alpha/2,n-1} \leq T_0 \leq t_{\alpha/2,n-1} \text{ when } \delta \neq 0\}$$
$$= P\{-t_{\alpha/2,n-1} \leq T_0' \leq t_{\alpha/2,n-1}\}$$

Fortunately, this unpleasant task has already been done, and the results are summarized in a series of graphs in Appendix A Charts Va, Vb, Vc, and Vd that plot for the *t*-test against a parameter δ for various sample sizes *n*.

4-5 Inference on the Mean of a Population, Variance Unknown

4-5.2 Type II Error and Choice of Sample Size

These graphics are called **operating characteristic** (or **OC**) **curves**. Curves are provided for two-sided alternatives on Charts Va and Vb. The abscissa scale factor *d* on these charts is defined as

$$d = \frac{|\mu - \mu_0|}{\sigma} = \frac{|\delta|}{\sigma}$$

4-5 Inference on the Mean of a Population, Variance Unknown

4-5.3 Confidence Interval on the Mean

Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal distribution with unknown variance σ^2, a $100(1 - \alpha)\%$ CI on μ is given by

$$\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n} \leq \mu \leq \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n} \quad (4\text{-}50)$$

where $t_{\alpha/2,n-1}$ is the upper $100\alpha/2$ percentage point of the t distribution with $n - 1$ degrees of freedom.

4-5 Inference on the Mean of a Population, Variance Unknown

4-5.3 Confidence Interval on the Mean

EXAMPLE 4-9

Golf Clubs

Reconsider the golf club coefficient of restitution problem in Example 4-7. We know that $n = 15$, $\bar{x} = 0.83725$, and $s = 0.02456$. Find a 95% CI on μ.

Solution. From equation 4-50 we find ($t_{\alpha/2,n-1} = t_{0.025,14} = 2.145$):

$$\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n} \leq \mu \leq \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n}$$

$$0.83725 - 2.145(0.02456)/\sqrt{15} \leq \mu \leq 0.83725 + 2.145(0.02456)/\sqrt{15}$$

$$0.83725 - 0.01360 \leq \mu \leq 0.83725 + 0.01360$$

$$0.82365 \leq \mu \leq 0.85085$$

4-5 Inference on the Mean of a Population, Variance Unknown

4-5.4 Confidence Interval on the Mean

EXAMPLE 4-9

In Example 4-7, we tested a one-sided alternative hypothesis on μ. Some engineers might be interested in a one-sided confidence bound. Recall that the Minitab output actually computed a lower confidence bound. The 95% lower confidence bound on mean coefficient of restitution is

$$\bar{x} - t_{0.05,n-1}s/\sqrt{n} \leq \mu$$
$$0.83725 - 1.761(0.02456)/\sqrt{15} \leq \mu$$
$$0.82608 \leq \mu$$

Thus we can state with 95% confidence that the mean coefficient of restitution exceeds 0.82608. This is also the result reported by Minitab. ■

4-6 Inference on the Variance of a Normal Population

4-6.1 Hypothesis Testing on the Variance of a Normal Population

$$H_0\colon \sigma^2 = \sigma_0^2$$
$$H_1\colon \sigma^2 \neq \sigma_0^2$$

$$X_0^2 = \frac{(n-1)S^2}{\sigma_0^2} \tag{4-52}$$

4-6 Inference on the Variance of a Normal Population

4-6.1 Hypothesis Testing on the Variance of a Normal Population

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. The quantity

$$X^2 = \frac{(n-1)S^2}{\sigma^2} \qquad (4\text{-}53)$$

has a chi-square distribution with $n - 1$ degrees of freedom, abbreviated as χ^2_{n-1}. In general, the probability density function of a chi-square random variable is

$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{(k/2)-1} e^{-x/2} \qquad x > 0 \qquad (4\text{-}54)$$

where k is the number of degrees of freedom and $\Gamma(k/2)$ was defined in Section 4-5.1.

4-6 Inference on the Variance of a Normal Population

4-6.1 Hypothesis Testing on the Variance of a Normal Population

The mean and variance of the χ^2 distribution are

$$\mu = k \qquad \text{and} \qquad \sigma^2 = 2k$$

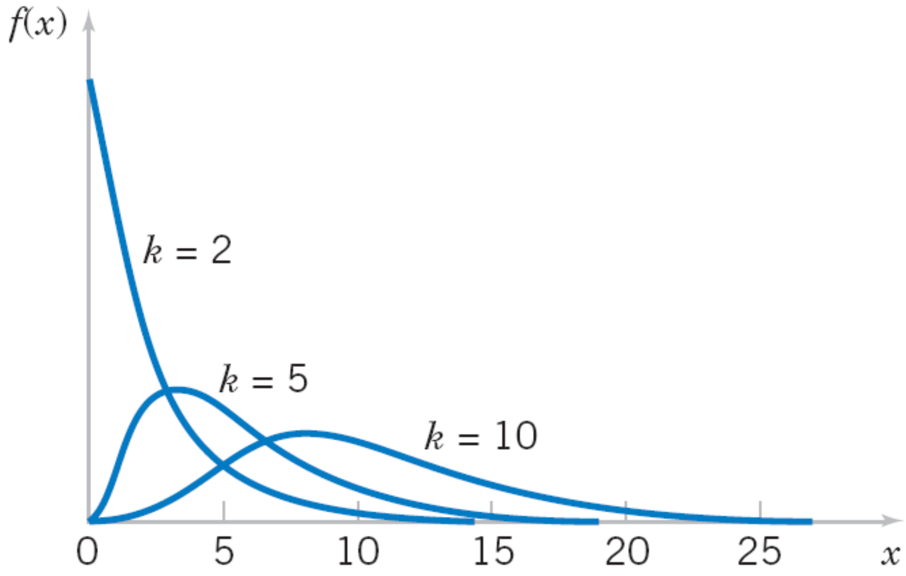


Figure 4-21 Probability density functions of several χ^2 distributions.

4-6 Inference on the Variance of a Normal Population

4-6.1 Hypothesis Testing on the Variance of a Normal Population

$$P(X^2 > \chi^2_{\alpha,k}) = \int_{\chi^2_{\alpha,k}}^{\infty} f(u)\,du = \alpha$$

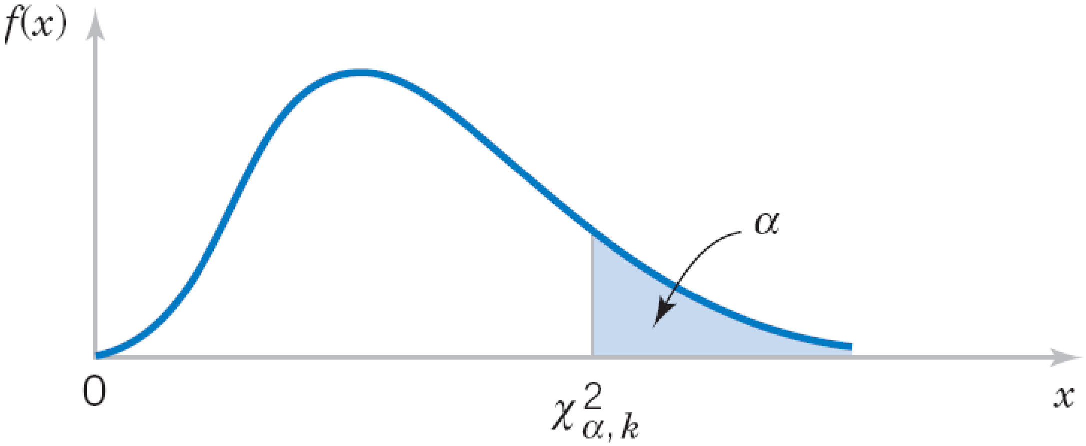


Figure 4-22 Percentage point $\chi^2_{\alpha,k}$ of the χ^2 distribution.

4-6 Inference on the Variance of a Normal Population

4-6.1 Hypothesis Testing on the Variance of a Normal Population

Testing Hypotheses on the Variance of a Normal Distribution

Null hypothesis: $H_0\colon \sigma^2 = \sigma_0^2$

Test statistic: $\chi_0^2 = \dfrac{(n-1)S^2}{\sigma_0^2}$

Alternative Hypotheses	Rejection Criterion
$H_1\colon \sigma^2 \neq \sigma_0^2$	$\chi_0^2 > \chi_{\alpha/2,n-1}^2$ or $\chi_0^2 < \chi_{1-\alpha/2,n-1}^2$
$H_1\colon \sigma^2 > \sigma_0^2$	$\chi_0^2 > \chi_{\alpha,\,n-1}^2$
$H_1\colon \sigma^2 < \sigma_0^2$	$\chi_0^2 < \chi_{1-\alpha,\,n-1}^2$

The locations of the critical region are shown in Fig. 4-23.

4-6 Inference on the Variance of a Normal Population

4-6.1 Hypothesis Testing on the Variance of a Normal Population

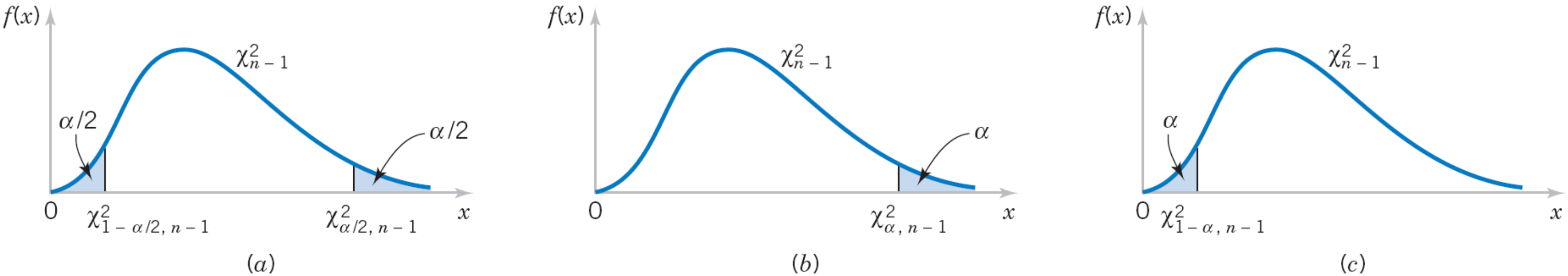


Figure 4-23 Distribution of the test statistic for H_0: $\sigma^2 = \sigma_0^2$ with critical region values for (a) H_1: $\sigma^2 \neq \sigma_0^2$, (b) H_0: $\sigma^2 > \sigma_0^2$, and (c) H_0: $\sigma^2 < \sigma_0^2$.

4-6 Inference on the Variance of a Normal Population

4-6.2 Confidence Interval on the Variance of a Normal Population

Confidence Interval on the Variance of a Normal Distribution

If s^2 is the sample variance from a random sample of n observations from a normal distribution with unknown variance σ^2, a $100(1 - \alpha)\%$ CI on σ^2 is

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}} \qquad (4\text{-}62)$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the upper and lower $100\alpha/2$ percentage points of the chi-square distribution with $n - 1$ degrees of freedom, respectively.

4-7 Inference on Population Proportion

4-7.1 Hypothesis Testing on a Binomial Proportion

We will consider testing:

$$H_0: p = p_0$$
$$H_1: p \neq p_0$$

Let X be the number of observations in a random sample of size n that belongs to the class associated with p. Then the quantity

$$Z = \frac{X - np}{\sqrt{np(1 - p)}} \tag{4-64}$$

has approximately a standard normal distribution, $N(0, 1)$.

4-7 Inference on Population Proportion

4-7.1 Hypothesis Testing on a Binomial Proportion

Testing Hypotheses on a Binomial Proportion

Null hypotheses: $H_0\colon p = p_0$

Test statistic: $$Z_0 = \frac{X - np_0}{\sqrt{np_0(1 - p_0)}}$$

Alternative Hypotheses	*P*-Value	Rejection Criterion for Fixed-Level Tests
$H_1\colon p \neq p_0$	Probability above z_0 and probability below $-z_0$, $P = 2\left[1 - \Phi(\lvert z_0 \rvert)\right]$	$z_0 > z_{\alpha/2}$ or $z_0 < -z_{\alpha/2}$
$H_1\colon p > p_0$	Probability above z_0, $P = 1 - \Phi(z_0)$	$z_0 > z_\alpha$
$H_1\colon p < p_0$	Probability below z_0, $P = \Phi(z_0)$	$z_0 < -z_\alpha$

4-7 Inference on Population Proportion

4-7.1 Hypothesis Testing on a Binomial Proportion

EXAMPLE 4-12

Engine Controllers

A semiconductor manufacturer produces controllers used in automobile engine applications. The customer requires that the process fallout or fraction defective at a critical manufacturing step not exceed 0.05 and that the manufacturer demonstrate process capability at this level of quality using $\alpha = 0.05$. The semiconductor manufacturer takes a random sample of 200 devices and finds that 4 of them are defective. Can the manufacturer demonstrate process capability for the customer?

Solution. We may solve this problem using the seven-step hypothesis testing procedure as follows:

1. **Parameter of interest:** The parameter of interest is the process fraction defective p.
2. **Null hypothesis, H_0:** $p = 0.05$
3. **Alternative hypothesis, H_1:** $p < 0.05$
   This formulation of the problem will allow the manufacturer to make a strong claim about process capability if the null hypothesis H_0: $p = 0.05$ is rejected.

4-7 Inference on Population Proportion

4-7.1 Hypothesis Testing on a Binomial Proportion

EXAMPLE 4-12

4. **Test statistic:** The test statistic is (from equation 4-64)

$$z_0 = \frac{x - np_0}{\sqrt{np_0(1 - p_0)}}$$

where $x = 4$, $n = 200$, and $p_0 = 0.05$.

5. **Reject H_0 if:** Reject H_0: $p = 0.05$ if the P-value is less than 0.05.

6. **Computations:** The test statistic is

$$z_0 = \frac{4 - 200(0.05)}{\sqrt{200(0.05)(0.95)}} = -1.95$$

7. **Conclusions:** Because $z_0 = -1.95$, the P-value is $\Phi(-1.95) = 0.0256$; since this is less than 0.05, we reject H_0 and conclude that the process fraction defective p is less than 0.05. We conclude that the process is capable. ■

4-7 Inference on Population Proportion

4-7.2 Type II Error and Choice of Sample Size

The approximate β-error for the two-sided alternative H_1: $p \neq p_0$ is

$$\beta = \Phi\left[\frac{p_0 - p + z_{\alpha/2}\sqrt{p_0(1 - p_0)/n}}{\sqrt{p(1 - p)/n}}\right] - \Phi\left[\frac{p_0 - p - z_{\alpha/2}\sqrt{p_0(1 - p_0)/n}}{\sqrt{p(1 - p)/n}}\right] \quad (4\text{-}66)$$

If the alternative is H_1: $p < p_0$,

$$\beta = 1 - \Phi\left[\frac{p_0 - p - z_{\alpha}\sqrt{p_0(1 - p_0)/n}}{\sqrt{p(1 - p)/n}}\right] \quad (4\text{-}67)$$

whereas if the alternative is H_1: $p > p_0$,

$$\beta = \Phi\left[\frac{p_0 - p + z_{\alpha}\sqrt{p_0(1 - p_0)/n}}{\sqrt{p(1 - p)/n}}\right] \quad (4\text{-}68)$$

4-7 Inference on Population Proportion

4-7.2 Type II Error and Choice of Sample Size

Sample Size for a Two-Sided Hypothesis Test on a Binomial Proportion

$$n = \left[\frac{z_{\alpha/2}\sqrt{p_0(1 - p_0)} + z_{\beta}\sqrt{p(1 - p)}}{p - p_0} \right]^2 \qquad (4\text{-}69)$$

If n is not an integer, round the sample size up to the next larger integer.

4-7 Inference on Population Proportion

4-7.3 Confidence Interval on a Binomial Proportion

Confidence Interval on a Binomial Proportion

If $\hat{p}$ is the proportion of observations in a random sample of size n that belong to a class of interest, an approximate $100(1 - \alpha)\%$ CI on the proportion p of the population that belongs to this class is

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad (4\text{-}73)$$

where $z_{\alpha/2}$ is the upper $100\,\alpha/2$ percentage point of the standard normal distribution.

4-7 Inference on Population Proportion

4-7.3 Confidence Interval on a Binomial Proportion

EXAMPLE 4-14

Crankshaft Bearings

In a random sample of 85 automobile engine crankshaft bearings, 10 have a surface finish that is rougher than the specifications allow. Find a 95% confidence interval on the proportion of defective bearings.

Solution. A point estimate of the proportion of bearings in the population that exceeds the roughness specification is $\hat{p} = x/n = 10/85 = 0.12$. A 95% two-sided CI for p is computed from equation 4-73 as

$$\hat{p} - z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

or

$$0.12 - 1.96\sqrt{\frac{0.12(0.88)}{85}} \leq p \leq 0.12 + 1.96\sqrt{\frac{0.12(0.88)}{85}}$$

which simplifies to

$$0.05 \leq p \leq 0.19$$

4-7 Inference on Population Proportion

4-7.3 Confidence Interval on a Binomial Proportion

Choice of Sample Size

Sample Size for a Specified E on a Binomial Proportion

If $\hat{P}$ is used as an estimate of p, we can be $100(1-\alpha)\%$ confident that the error $|\hat{P}-p|$ will not exceed a specified amount E when the sample size is

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 p(1-p) \tag{4-74}$$

For a specified error E, an upper bound on the sample size for estimating p is

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 \frac{1}{4} \tag{4-75}$$

4-8 Other Interval Estimates for a Single Sample

4-8.1 Prediction Interval

A **$100(1 - \alpha)\%$ PI on a single future observation from a normal distribution** is given by

$$\bar{x} - t_{\alpha/2,n-1}s\sqrt{1 + \frac{1}{n}} \le X_{n+1} \le \bar{x} + t_{\alpha/2,n-1}s\sqrt{1 + \frac{1}{n}} \qquad (4\text{-}76)$$

4-8 Other Interval Estimates for a Single Sample

4-8.2 Tolerance Intervals for a Normal Distribution

A **tolerance interval** to contain at least $\gamma\%$ of the values in a normal population with confidence level $100(1-\alpha)\%$ is

$$\bar{x} - ks, \bar{x} + ks$$

where k is a tolerance interval factor for the normal distribution found in Appendix A Table VI. Values of k are given for $1 - \alpha = 0.90, 0.95, 0.99$ confidence level and for $\gamma = 90, 95$, and 99%.

4-10 Testing for Goodness of Fit

- So far, we have assumed the population or probability distribution for a particular problem is known.
- There are many instances where the underlying distribution is not known, and we wish to test a particular distribution.
- Use a goodness-of-fit test procedure based on the chi-square distribution.

$$X_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \tag{4-77}$$

IMPORTANT TERMS AND CONCEPTS

- Alternative hypothesis
- Bias in estimation
- Chi-squared distribution
- Comparative experiment
- Confidence bound
- Parameter estimation
- Point estimation
- Power of a test
- Practical significance versus statistical significance
- Precision of estimation
- Prediction interval
- Confidence coefficient
- Confidence interval
- Confidence level
- Confidence limits
- Coverage
- Critical region
- Estimated standard error
- Probability of a type I error
- Probability of a type II error
- Procedure for hypothesis testing
- Relative efficiency of an estimator
- Fixed significance level hypothesis testing
- Goodness of fit
- Hypothesis testing
- Minimum variance unbiased estimator
- Null hypothesis
- Sample size determination
- Significance level
- Standard error
- Statistical hypothesis
- Statistical inference
- *t*-distribution
- One-sided alternative hypothesis
- One-sided confidence bounds
- Operating characteristic curves
- *P*-values
- Test statistic
- Tolerance interval
- Two-sided alternative hypothesis
- Type I error
- Type II error